More constraints in Flag Algebras

Last time, we explored how to solve a problem of the following type. Let G be a fixed graph.

$$\text{(Problem)} \begin{cases} \text{Minimize} & t \\ \text{subject to} & \phi(G) \leq t \\ & \text{for all } \phi \in Hom^+(\mathcal{A}, \mathbb{R}) \end{cases}$$

Recall that $\mathcal{A}$ might not be all graphs, but for example just triangle-free graphs.

If one can prove some special property about the extremal examples, even without knowing them, it may be utilized by flag algebras. We will try an example using the Mantel's theorem again. Here is the property.

1: Let G be a triangle-free graph maximizing the number of edges. Show that for any two vertices u and v in $V(G)$, their degree is different by at most one.

Solution: Suppose $d(u) - d(v) \geq 2$. Let G' be obtained from G by removing v and adding a duplicate of u, called u'. The edge uu' is not present in G. If u and v are adjacent, then

$$|E(G')| - |E(G)| = 2(d(u) - 1) - (d(u) + d(v) - 1) = d(u) - d(v) - 1 \geq 1.$$

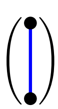
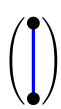
If u and v are not adjacent, then

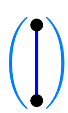
$$|E(G')| - |E(G)| = 2d(u) - (d(u) + d(v)) = d(u) - d(v) \geq 2.$$

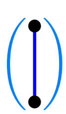
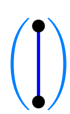
We also need to show, that G' is triangle free. Suppose for contradiction that T is a triangle in G'. It cannot contain both u and u' since they are not adjacent. If T contains u', it can be swapped for u, since u and u' have the same neighborhood. Therefore, T exists in G, which is a contradiction.
Finally, G' is a triangle-free graph with more edges than G, contradiction.

Suppose that $(G_n)_{n \in \mathbb{N}}$ is a convergent sequence such that G_n is an extremal graph on n vertices. Then there is a corresponding $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$. Lets consider only such homomorphism and call the set $ExHom^+(\mathcal{A}, \mathbb{R})$ as Extremal Homomorphisms.

Let $(G_n)_{n \in \mathbb{N}}$ be a convergent sequence, where G_n is a complete bipartite graph on n vertices. Let $\phi_B \in Hom^+(\mathcal{A}, \mathbb{R})$ correspond to this bipartite sequence. Suppose we do not know it is an extremal sequence yet.

2: What can you say about ϕ_B  and what does it mean for ϕ' , where $\phi' \in ExHom^+(\mathcal{A}, \mathbb{R})$?

Solution: We now know that ϕ_B  $= \frac{1}{2}$. This gives a lower bound on the extremal homomorphisms, so

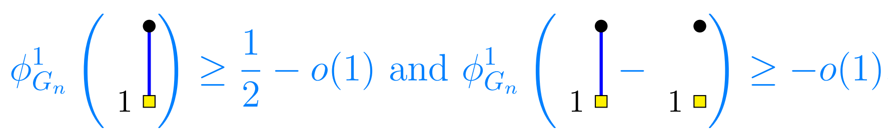
ϕ'  $\geq \phi_B$  $= \frac{1}{2}$ for all $\phi' \in ExHom^+(\mathcal{A}, \mathbb{R})$.

Let 1 be a graph containing exactly one vertex. Let $\mathcal{A}^1$ be a flag algebra with one labeled vertex.

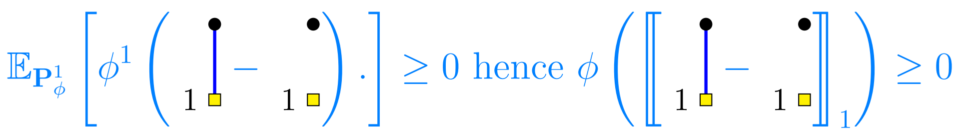
3: Let G_n be an extremal graph. Show that for all $\phi^1_{G_n} \in \mathbf{P}^1_{G_n}$.

Solution: Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of extremal graphs.

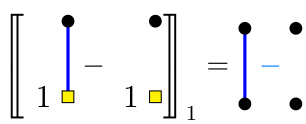
Since all vertices have the same degree and the density of edges is at least $1/2$, we get that the degree divided by $n-1$ is at least $1/2 - o(1)$. This is also equivalent to say that a vertex is incident to more edges than non-edges.



Consider $\phi \in ExHom^+(\mathcal{A}, \mathbb{R})$ as a limit of a convergent sequence $(G_n)_{n\in\mathbb{N}}$ be of extremal graphs. By the weak convergence of $\mathbf{P}^1_{G_n}$ to $\mathbf{P}^1_\phi$ we have



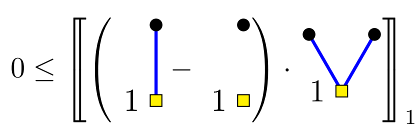
4: Expand the average



What is the derived inequality saying about $\phi \in ExHom^+(\mathcal{A}, \mathbb{R})$? It is saying there are more edges than nonedges. We already knew this!

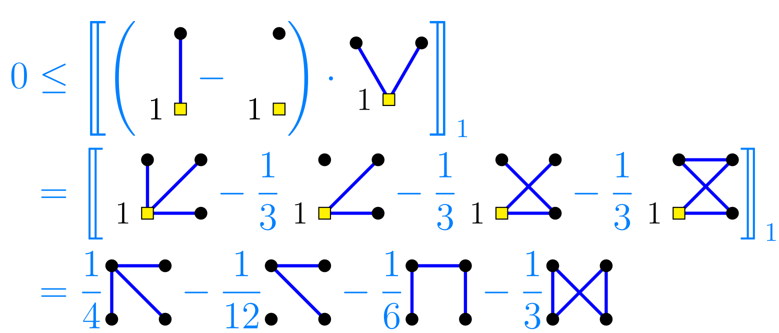
Now for the additional constraint, we use a small trick that if $\phi^1(A) \geq 0$ and $\phi^1(B) \geq 0$, then also $\phi^1(A \cdot B) \geq 0$.

5: Expand the right hand side as a linear combination of triangle-free graphs.



(1)

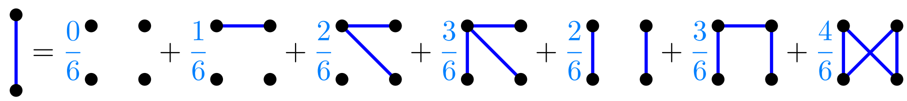
Solution:



Notice that when we use (1), we can use any positive multiple of it.

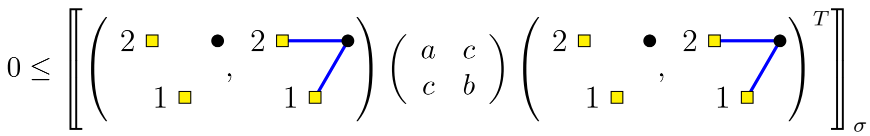
Now we try to show it can be used to prove Mantel's theorem. It could be done by combining together (2), (3), and (1).

6: Fill coefficients on the right hand side. (triangle-free)



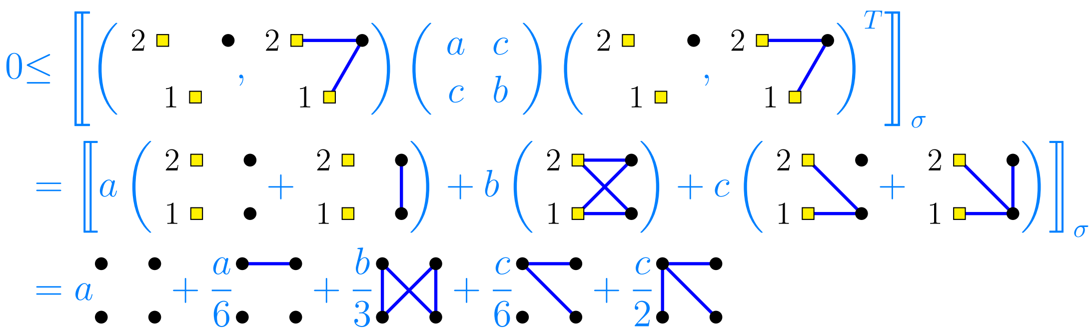
(2)

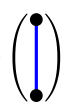
7: Calculate the right hand side as a linear combination of unlabeled (triangle-free) graphs. What is σ?

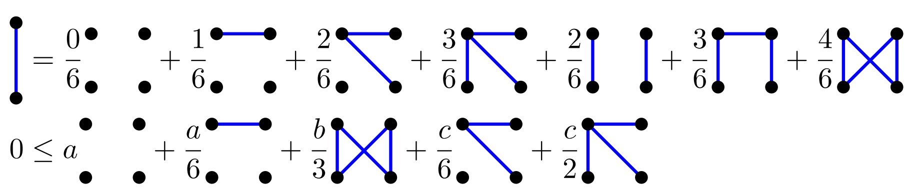


(3)

Solution: $\sigma =$

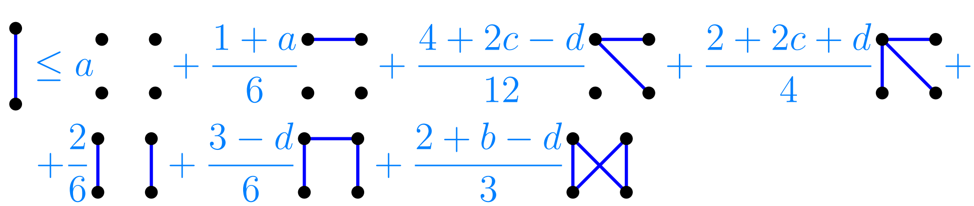


8: Write a semidefinite program for finding an upper bound on ϕ  and guess its solution. Do it by combining (2), (3) and take d times (1), where $d \geq 0$ is a variable.
Expansions of (2) and (3) from previous questions are below for your convenience.



Hint: First write $K_2 \leq \sum_{F \in \mathcal{F}_4} c_F \cdot F$, where c_F depends on a, b, c, d, and then make the SDP.

Solution: By combining (2), (3), and (1), where (1) is with coefficient d, we get



(4)

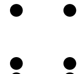
(5)

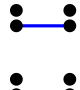
And goal is to show $\phi\Big($  $\Big) \leq \frac{1}{2}$ The SDP then looks like

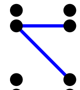
(SDP)

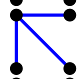
Minimize t

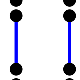
subject to

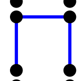
 : $a \leq t$

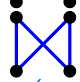
 : $(1+a)/6 \leq t$

 : $(4+2c-d)/12 \leq t$

 : $(2+2c+d)/4 \leq t$

 : $2/6 \leq t$

 : $(3-d)/6 \leq t$

 : $(2+b-d)/3 \leq t$

$$\begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & t \end{pmatrix} \succcurlyeq 0$$

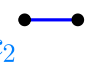
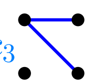
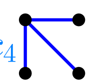
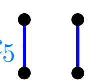
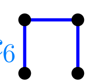
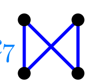
Solution that works is

$$a = \frac{1}{2}, b = \frac{1}{2}, c = -\frac{1}{2}, d = 1, t = \frac{1}{2}.$$

Notice it would be possible to multiply (4) by 12 and then the resulting (SDP) has no fractions, while the optimum would be 6 instead of $\frac{1}{2}$. We will use this later.

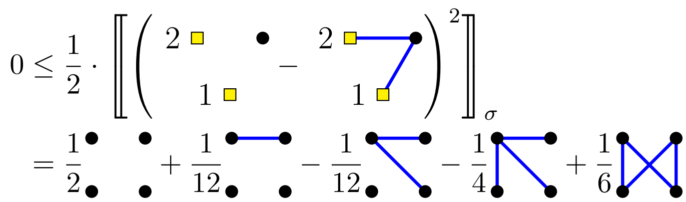
9: Which of the constraints are tight? Can you explain why they must be tight?
Hint: What is the extremal example?

Solution: By combining equation (2), (3), and (1), we proved there exist $c_1, \ldots, c_7$ such that

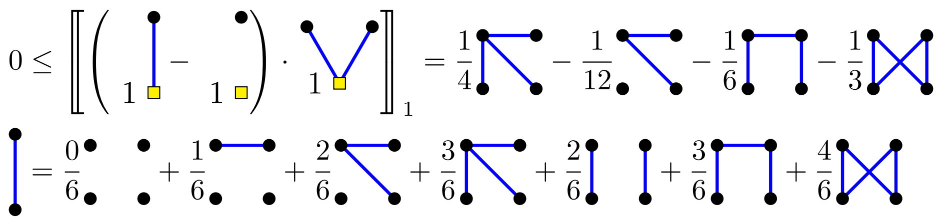
 $= c_1$  $+ c_2$  $+ c_3$  $+ c_4$  $+ c_5$  $+ c_6$  $+ c_7$  $\leq \max_i \{c_i\} = \frac{1}{2}.$

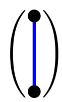
Let the graph with coefficient c_i be G_i. Imagine an extremal $\phi \in Hom^+(\mathcal{A}^1, \mathbb{R})$, where $\phi\Big($  $\Big) = \frac{1}{2}$. That means the inequality $\leq \max_i\{c_i\}$ must be tight. Hence for any G, if $\phi(G_i) > 0$, then $c_i = \frac{1}{2}$. Otherwise the linear combination $\phi(\sum_i c_i G_i)$ would sum up to something smaller than $\frac{1}{2}$.

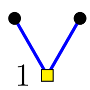
The previous solution can be rewritten even slightly easier. Notice that the matrix $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ in the optimal solution has $\frac{1}{2} = a = b = -c$. This gives eigenvectors $(1,1)$ and $(1,-1)$, where $(1,1)$ corresponds to zero eigenvalue. We can then use just one square



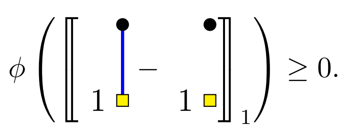
instead of (3). For convenience, we repeat the following



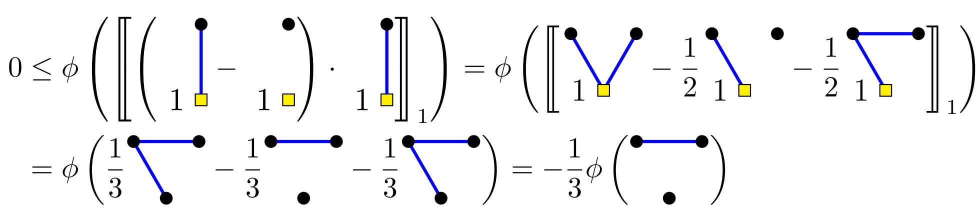
By summing them up, we conclude ϕ  $\leq \frac{1}{2}$.

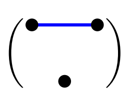
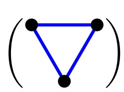
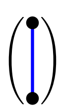
One may ask, why did we pick  for the multiplication. Well, we tried all possibilities and this one worked.

Mantel Once Again! This time with a trick proof. Let $\phi \in ExHom^+(\mathcal{A}, \mathbb{R})$. We have already proved that



This implies



Therefore, we get ϕ  $= \phi$  $= 0$. This implies, that the sequences for ϕ are *close* to sequences of complete bipartite graphs. And this in turn gives ϕ  $\leq \frac{1}{2}$.